Self-erasing perturbations of Abelian sandpiles

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We investigate generalized seeding of the attracting states of Abelian sandpile automata and find there exists a class of global perturbations of such automata that are completely removed by the natural local dynamics. We derive a general form for such *self-erasing perturbations* and demonstrate that they can be highly nontrivial. This phenomenon provides a different conceptual framework for studying such automata and suggests possible applications for data protection and encryption.

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I. INTRODUCTION

In 1987, Bak, Tang, and Wiesenfeld (BTW) [1] introduced the scalar sandpile automaton as a prototype for self-organized criticality [2], a possible mechanism for the spontaneous emergence of temporal and spatial scale invariance in many-body systems. While the sandpile automaton may not be a good model of real sandpiles, it is a useful model of nonequilibrium statistical mechanics, with many exactly solvable features, and is related to important problems, including resistor networks, the Potts model, and loop-erased random walks [3]. Thus an understanding of such an automaton may have implications for a broad range of physical phenomena. Reflecting this potential, the canonical model introduced by BTW has been studied extensively and generalized in many ways [4]. In one such generalization, Dhar [5] developed a subclass of scalar sandpiles, which he termed Abelian, because positive point perturbations of the piles—the additions of single grains of sand—commute. In fact, the idea of perturbing such automata with positive point perturbations has been widely employed, as it aptly and simply illustrates the emergence of global effects on the pile from the operation of the local rules at each site. While these perturbations have typically been applied randomly, interesting effects emerge even if grains are added deterministically, for example, only to the center site [6].

In this paper, we study the consequences of perturbing Abelian sandpiles in more complicated ways, including those involving the concurrent addition *and* subtraction of *multiple* grains at *multiple* sites. In particular, we study one specific emergent phenomenon of such generalized "seeding." Namely, we investigate a class of nontrivial *global* perturbations of attracting piles that are completely removed by the natural *local* dynamics of the piles. We derive a general form for these *self-erasing perturbations* (SEP's), employ them to better understand the behaviors of the sandpiles, and illustrate how such perturbations might be used to protect or encrypt information.

The paper is organized in the following way. Section II motivates and derives the Abelian restriction on the sandpile rules in prelude to the derivation of a general form for SEP's on Abelian sandpile attractors in Sec. III. Section IV demonstrates an *inward shift invariance* of BTW sandpile attractors, a special and provocative example of SEP's. Sections V and VI suggest the possible ways in which this phenomenon may be applied towards both a general understanding of the behavior of Abelian sandpiles and possible applications for data protection and encryption. Section VII suggests the generality of this phenomenon by introducing a related automaton that displays a similar behavior. We summarize our conclusions in Sec. VIII.

II. ABELIAN RULES

A. Toppling matrix

Formally, we represent a sandpile of any spatial dimension as a row vector $\mathbf{P}=\{P_1,P_2,\ldots,P_N\}$ whose ith column contains the number of grains (or height of the pile) P_i at the ith site of the N-site automaton. If the number of grains at the ith site exceeds a critical threshold $P_i > C_i$, which may differ from site to site, it *topples*, so that grains are redistributed (or added or deleted) according to certain toppling rules. Following Dhar [5], we store the number of grains to be added or subtracted when the ith site topples in the ith row of a square matrix Δ. For example, the toppling rules for a four-site one-dimensional BTW sandpile automaton are stored in the 4×4 tridiagonal matrix,

$$\Delta_{\mathrm{BTW}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \tag{1}$$

We define the *relaxation* of a pile to be the successive toppling of all supercritical sites until every site is subcritical. Using the unit row vector $\mathbf{e}_i$ to select the appropriate rule, we denote the toppling of the ith site by

$$T(\mathbf{P}) = \mathbf{P} - \mathbf{e}_i\Delta, \tag{2}$$

and the relaxation of the pile by

$$R(\mathbf{P}) = \mathbf{P} - \mathbf{n}\Delta = \mathbf{A}, \tag{3}$$

where $\mathbf{n}$ is a row vector whose ith column contains the number of topples n_i occurring at the ith site during the relaxation of the supercritical pile $\mathbf{P}$ to the subcritical pile $\mathbf{A}$.

B. Abelian constraints

For general Δ, a supercritical state may relax to a stable subcritical pile, or diverge to infinity, or cycle among multiple configurations. Furthermore, the outcome may depend on the *order* in which the supercritical sites are toppled. For example, synchronous toppling and random toppling may relax a supercritical pile to different subcritical piles. It is crucial for what follows that any supercritical state relaxes to a subcritical state *independent* of toppling order. Consequently, we constrain Δ to guarantee this Abelian property.

First, in order for a supercritical pile to become subcritical, toppling a supercritical site must tend to remove grains from that site. Hence we require

$$\Delta_{ii} > 0. \tag{4a}$$

Also, because the total number of grains in a supercritical pile is greater than the total number of grains in a subcritical pile, the toppling rules must have a way to remove grains. We require that at each site the rule be either conservative or dissipative (and not generative),

$$\sum_j \Delta_{ij} \geqslant 0, \tag{4b}$$

while we demand that *overall* the toppling rules be dissipative,

$$\sum_i \sum_j \Delta_{ij} > 0. \tag{4c}$$

Next, we must avoid oscillations in the relaxation. These can occur if, during the relaxation, some pile $\mathbf{P}'$ occurs multiple times. In analogy with Eq. (3), this implies that $\mathbf{P}' = \mathbf{P}' - \mathbf{n}\Delta$ or $\mathbf{n}\Delta = \mathbf{0}$. However, if the columns of Δ are linearly independent, this equation will only have the trivial solution, and oscillations will be prevented. Equivalently, we require that

$$\det(\Delta) \neq 0. \tag{4d}$$

The restrictions of Eq. (4) are sufficient to ensure that a supercritical pile relaxes to a stable subcritical pile [7].

Finally, in order for the relaxation to be robust with respect to the order of toppling, toppling at one site cannot interfere with the possibility of toppling at another site. Thus we require that toppling at one site does not remove grains from other sites, and our final condition on Δ is

$$\Delta_{ij} \leqslant 0, \; j \neq i. \tag{5}$$

These constraints [8] are equivalent to those introduced by Dhar [5] for the Abelian class, and we hereafter limit our discussion to Abelian sandpiles.

III. SELF-ERASING PERTURBATIONS

Under these conditions, define an *attractor* to be a subcritical pile relaxed from a supercritical pile [9]. In general, adding grains to an attractor and then relaxing will produce another attractor while subtracting grains risks creating a state to which no supercritical pile will relax. However, subtractions made concurrently with additions may still allow the pile to relax to another attractor. More strongly, certain such "mixed" perturbations may actually map an attractor back to itself. These are the perturbations that we have termed self-erasing because the natural relaxation of the pile completely removes all trace of them. From Eq. (3), if $R(\mathbf{A} + \delta\mathbf{A}) = \mathbf{A}$, then $\delta\mathbf{A} = \mathbf{n}\Delta$. Thus all self-erasing perturbations are whole number combinations of the toppling rules of the sandpile.

However, for Abelian sandpiles, it is also conversely true that every perturbation of this form is self-erasing. To demonstrate this, we recognize that for every attractor $\mathbf{A}$, there is a supercritical pile $\mathbf{P}$ such that $R(\mathbf{P}) = \mathbf{A}$. We then perturb $\mathbf{P}$ with $\mathbf{n}\Delta$, where $n_i \geqslant 0$. Because the sandpile is Abelian, any toppling order will produce the same final relaxed state. In particular, we can choose to topple such that the added $\mathbf{n}\Delta$ is removed *first*, so that it relaxes to the intermediary pile $\mathbf{P}$, $R(\mathbf{P} + \mathbf{n}\Delta) = R(\mathbf{P})$. Alternately, we can choose to topple such that the added $\mathbf{n}\Delta$ is initially ignored and let it "ride down" as the rest of the pile relaxes to the original subcritical pile, $R(\mathbf{P} + \mathbf{n}\Delta) = R(\mathbf{A} + \mathbf{n}\Delta)$. Combining these results, we get $R(\mathbf{A} + \mathbf{n}\Delta) = R(\mathbf{P} + \mathbf{n}\Delta) = R(\mathbf{P}) = \mathbf{A}$. Hence, for all attractors $\mathbf{A}$,

$$R(\mathbf{A} + \delta\mathbf{A}) = \mathbf{A}, \tag{6a}$$

for all perturbations $\delta\mathbf{A}$ such that

$$\delta\mathbf{A} = \mathbf{n}\Delta, \; n_i \geqslant 0. \tag{6b}$$

Thus not only are all self-erasing perturbations whole number combinations of the toppling rules, but all such combinations are self-erasing perturbations. These self-erasing perturbations can be understood as dynamic realizations of the equivalence classes introduced by Dhar [5].

Although Eq. (6) may seem obvious in retrospect, it is in fact *false* for *most* toppling rules Δ and true for the relatively few toppling rules that are Abelian. Furthermore, as demonstrated below these perturbations can be quite nontrivial, involving large global changes to a pile. Removal of *global* perturbations by solely *local* rules is one of the distinctive features of this phenomenon.

Figure 1 illustrates the nontrivial nature of self-erasing perturbations for a one-dimensional pile relaxing according to the BTW rules. The grays code pile heights, and time increases downward. In the left panel (a), a supercritical pile relaxes to a subcritical pile. In the right panel (b), this attractor is seeded six times with different perturbations $\mathbf{n}\Delta$, and in each case the pile returns exactly to the original attractor. Note the complicated relaxation that is sometimes needed to remove these perturbations.

Since a finite-sized sandpile of any dimension can be indexed as a one-dimensional automaton with nonlocal communication among the sites, the above behavior will also exist in every automaton that satisfies the Abelian restrictions on Δ, regardless of the spatial orientation one assigns to the sites. As an example, in Fig. 2, we perturb a generic two-dimensional BTW attractor with a generic SEP $\mathbf{n}\Delta$ formed from a large random $\mathbf{n}$ with $\Sigma n_i = 10^4$. The perturbation relaxes leaving the attractor unchanged.

IV. INWARD SHIFT INVARIANCE

A special and provocative example of SEP's is *inward shifts*, a subclass of SEP's for two-dimensional BTW sand-

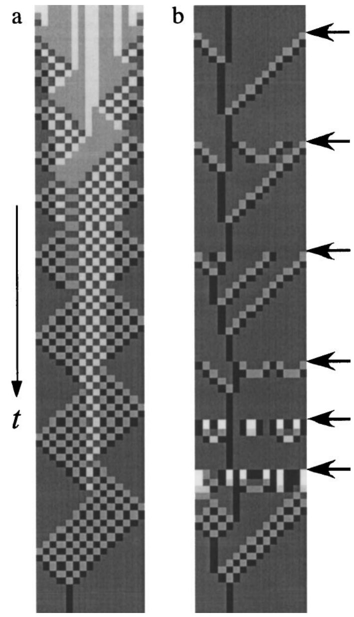

FIG. 1. (a) The relaxation of a supercritical one-dimensional sandpile according to the BTW rules. (b) The subsequent relaxations of six different self-erasing perturbations $\mathbf{n}\Delta$ (introduced at the arrows) to the same attractor. Grays code pile heights, and time increases downward.

piles. Inward shifts can be described simply yet demonstrate nicely the complicated spatiotemporal evolution that can accompany SEP's. They are special, simple cases of generic SEP $\mathbf{n}\Delta$, where $n_i=1$ for all sites inside a closed region and $n_i=0$ for all sites outside the region. Due to the specific form of the BTW rule, the perturbations to sites in the interior cancel, and the SEP can be generated by simply shifting grains one site inward across every segment of the boundary. Furthermore, such SEP can be superposed.

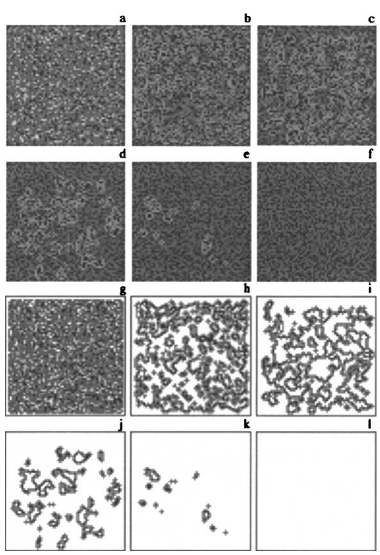

FIG. 2. Generic self-erasing perturbation $\mathbf{n}\Delta$ (with $\Sigma n_i=10^4$) of a generic attractor (formed by relaxing a generic supercritical 64×64 pile). The perturbations relax under two-dimensional BTW rules leaving the attractor unchanged. (a)–(f) Perturbation superimposed on the attractor. (g)–(l) Perturbation only.

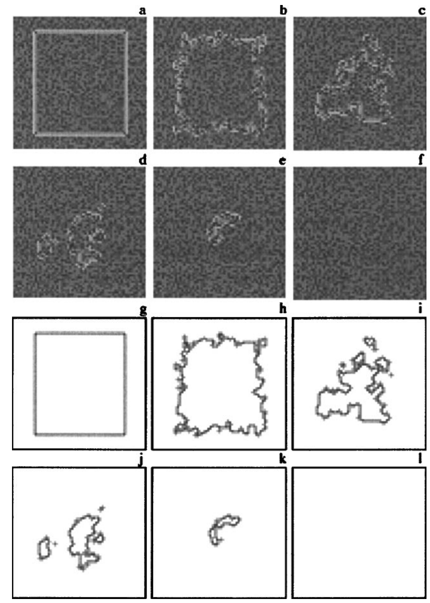

FIG. 3. Rectangular perturbation (formed by moving grains one site inward across a rectangular boundary) of a generic attractor (formed by relaxing a generic supercritical pile). The perturbations relax under two-dimensional BTW rules in a highly nontrivial way, leaving the attractor unchanged. Grays code heights and colors (online) code differences from original pile, with reds lower and blues higher. (a)–(f) Perturbation superimposed on the attractor. (g)–(l) Perturbation only.

Figures 3–5 illustrate inward shift invariance for three different perturbations of three qualitatively different attractors. Figure 3 summarizes the relaxation of a rectangular pertur-

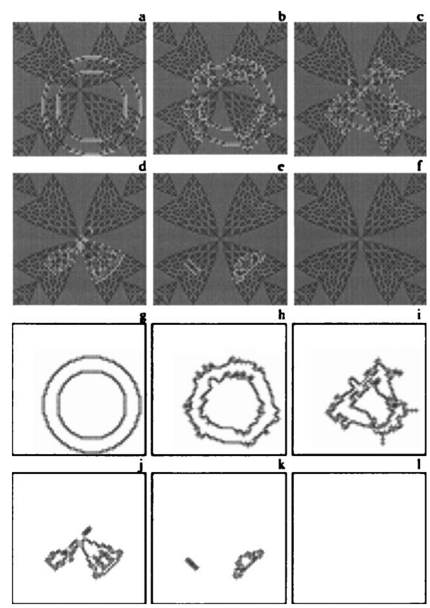

FIG. 4. Circular perturbations (formed by moving grains one site inward across all segments of circular boundaries) of a patterned attractor (formed by relaxing a flat supercritical pile). The perturbations relax under BTW rules leaving the attractor unchanged. (a)–(f) Perturbation superimposed on the attractor. (g)–(l) Perturbation only.

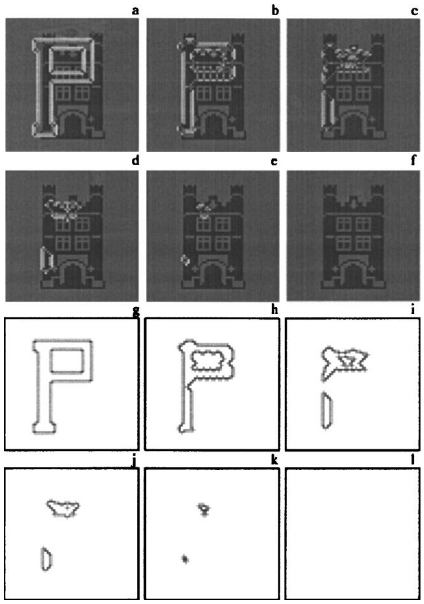

FIG. 5. "P"-shaped perturbation (formed by moving grains one site inward across the boundaries of the letter) of an image attractor (formed by using a black-and-white image as a mask to remove single grains from a critical pile). The perturbations relax under BTW rules leaving the attractor unchanged. (a)–(f) Perturbation superimposed on the attractor. (g)–(l) Perturbation only.

bation of a generic attractor, Fig. 4 summarizes the relaxation of a circular perturbation of a patterned attractor [10], and Fig. 5 summarizes the relaxation of a patterned perturbation of an image attractor. (As we further discuss in Sec. VI, any black-and-white image can be mapped into a canonical BTW attractor by using the image as a mask to remove a single layer of grains from a pile of critical height.) The complicated evolution that accompanies each relaxation is a hallmark of SEP's.

V. CONCEPTUAL BENEFITS

While SEP's supply a simple conceptual framework in which to understand phenomena like the inward shift invariance of the BTW attractors, SEP's also provide an alternative viewpoint for analyzing and elucidating the general behavior of Abelian sandpile automata. To demonstrate this, consider some notable results concerning such automata. Dhar [5] has demonstrated that the *number* of attracting piles is simply the determinant of the toppling matrix Δ, and that there exists some *Markov recurrence number* of grains that, when added to a single site of any attractor, will cause the pile to relax back to itself. Gabrielov [9] has discussed necessary *and* sufficient conditions on the toppling rules for a sandpile to be Abelian. Both Dhar [5] and Speer [11] have developed recursive algorithms to directly test if a given subcritical pile is an attractor. Using simple SEP arguments, we can readily recover many of these results.

In the SEP framework, single site Markov perturbations are just another specific type of SEP. Thus for a given Markov number m_i at a specific site i there exists $\mathbf{n}$ such that $\mathbf{n}\Delta = m_i\mathbf{e}_i$. This allows m_i to be found exactly. The Markov number m_i at site i is the smallest integer that solves $\mathbf{n} = m_i\mathbf{e}_i\Delta^{-1}$ such that all components of $\mathbf{n}$ are whole numbers. Because Δ is a matrix of whole numbers, its inverse can be decomposed into a determinant scaling factor and a matrix $\widetilde{\Delta}^{-1}$ whose elements are all integers. Thus this constraint becomes

$$\mathbf{n} = m_i\mathbf{e}_i\Delta^{-1} = \frac{m_i}{\det(\Delta)}\mathbf{e}_i\widetilde{\Delta}^{-1}. \tag{7}$$

Since the Markov number m_i is also the number of discrete states through which an evolution of the sandpile will pass, it is sufficient that $\det(\Delta)$ be the *maximum* number of intermediate states in the evolution, and hence the number of attracting states of the sandpile.

The SEP approach also provides useful insight into the development of necessary (as well as sufficient) conditions on the toppling rules to guarantee that the sandpile is well behaved and Abelian. According to Eq. (7), the *i*th column of $\Delta^{-1}$is simply the history vector for Markov addition at the *i*th site scaled by the corresponding Markov number, $\Delta^{-1} = [\mathbf{n}_1/m_1, \mathbf{n}_2/m_2, \ldots, \mathbf{n}_N/m_N]$. Thus a necessary condition for a sandpile to be well behaved and Abelian is clearly

$$\Delta^{-1}{}_{ij} \geqslant 0, \tag{8}$$

which supplements the work of Gabrielov [9].

SEP's also suggest a simple alternative to the attractor testing algorithms proposed by Dhar and Speer. Instead of recursively searching for forbidden subconfigurations [5] or "scripts" [11], one can exploit the fact that the SEP is a phenomenon of the attracting set only. For example, an effective attractor-testing algorithm might involve applying an SEP to a subcritical pile and allowing it to relax. If the relaxed pile is different from the original pile, then the original pile is not an attractor. Good test SEP's (such as the "covering" $\mathbf{n}\Delta$,where $n_i = 1$ for all i) would encourage toppling at every site.

However, if one wants to enumerate explicitly the set of attractors for a given toppling rule, testing individual subcritical piles with standard algorithms will become prohibitive as the pile size increases. Fortunately, since SEP's is a property of the attracting set only, by finding pile configurations that will cause simple SEP's to fail, one can identify broad classes of subcritical piles that are not attractors. For example, in the one-dimensional BTW sandpile, any SEP with $n_i = 1$ for two adjacent sites will fail on subcritical piles with $P_i = C_i - 1$ grains at those sites; thus any subcritical pile with such a configuration of pile heights will not be an attractor. For large sandpiles, enumerating attractors by eliminating those with characteristics that cause certain SEP's to fail may prove more efficient than recursive algorithms.

VI. APPLICATIONS

A. Encoding information

The remarkable resistance of Abelian sandpile attractors to a wide class of perturbations, as illustrated in Secs. III and IV, suggests several possible applications. We speculate on two possibilities here: the protection and encryption of data.

In both cases, information must be associated with attracting states. Furthermore, no matter how complicated or simple the desired method of encoding, it must address the restrictive nature of the attracting set in the choice of toppling rules.

Naively, one might expect that each site of a one-dimensional N-site BTW attractor could have either $C_i - 1$ or C_i grains, and a good encoding scheme might be to associate each site with a binary digit 0 or 1 of an N-digit binary word of data. However, simple SEP analysis demonstrates that the attractors are actually restricted to piles that have *no more than* one site with $C_i - 1$ grains (with all the rest having C_i grains). Thus instead of the 2^N attractors needed to encode an arbitrary N-digit binary word, a one-dimensional N-site BTW sandpile has only $N+1$ attractors. (While this restriction may complicate the association of information with attractors, it may also be desirable in certain applications.)

However, straightforward SEP analysis also demonstrates that two-dimensional N-site BTW sandpiles can encode any N-digit binary word in an N-site pile because all 2^N piles $\mathbf{B}$ with $B_i \in \{C_i - 1, C_i\}$ are indeed attractors. Furthermore, these rules can be modified to encode any level of information by appropriately scaling the toppling rules. If Δ_{BTW} represents the two-dimensional BTW rules and α is a whole number, then the attracting set of $\Delta' = \alpha \Delta_{\mathrm{BTW}}$ with $C_i' = \alpha C_i$ includes all $(2\alpha)^N$ piles $\mathbf{B}'$ with $B_i' \in \{\alpha C_i - 2\alpha + 1, \alpha C_i - 2\alpha + 2, \ldots, \alpha C_i\}$. Thus if the unscaled sandpile encodes two levels of information, then the scaled sandpile encodes 2α levels of information. In the following examples, we use the scaled two-dimensional BTW rules to encode a large range of pile heights as pixel grayscale values that compose images.

B. Data protection

The resilience of attractors to large perturbations suggests the possibility of exploiting SEP's to protect data. For example, if information were encoded in the attractor of a sandpile, and this sandpile were coupled to its environment *solely* via the addition of whole number combinations of the toppling rules $\mathbf{n}\Delta$, the information would be completely robust to all outside influence, including noise. In fact, given sufficient time, the natural dynamics would heal the perturbed attractor of all corruptions.

We demonstrate this self-healing property in Fig. 6, where the information of a 20-level gray scale image is encoded by manipulating the heights of the sites in the pile. To ensure that this is an attractor of our sandpile, we use the two-dimensional BTW rules scaled by a factor of $\alpha = 10$. Specifically, the brightness of each site in Fig. 6 is proportional to the number of grains at that site, resulting in a three-dimensional sculpture of the image. We then corrupt this image in the second frame (b) by adding a random whole number combination of scaled BTW rules. In the subsequent relaxation process (c)–(f), the sandpile completely heals itself of this corruption demonstrating the basic efficacy of this phenomenon.

C. Data encryption

This property also suggests a method of encryption. In Fig. 7, the same image has been corrupted, again by adding a random combination of scaled BTW rules. The perturbation is complicated enough to obscure the information stored in the attractor, thereby encrypting it. Decrypting this information is trivial if one knows the Δ with which the information was corrupted: simply relax the pile. However, without the proper Δ, relaxation does not restore the image. We illustrate this in Fig. 7 by relaxing the encrypted image with a *slightly* different Δ. (Instead of removing 40 grains during each topple, the toppling rule Δ' removes 41; this additional grain

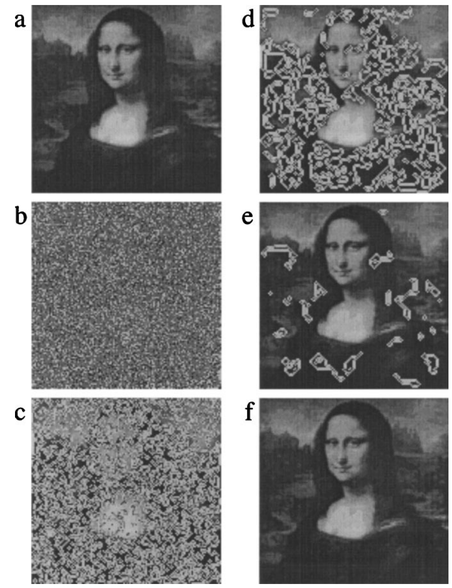

FIG. 6. (a) A grayscale image encoded, as a three-dimensional sculpture, in a 128×128 attractor. (b) The attractor corrupted by a random superposition $\mathbf{n}\Delta$ of scaled BTW rules. (c)–(f) The natural dynamics of the sandpile heals the corrupted attractor. Grays code heights and colors (online) code differences from original pile, with reds lower and blues higher.

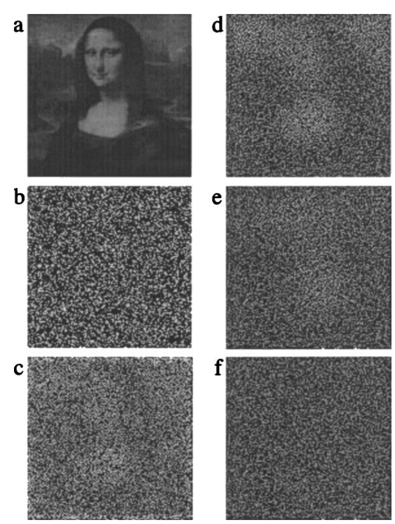

FIG. 7. (a) A grayscale image encoded in an attractor, as in Fig. 6. (b) The attractor corrupted by a random superposition $\mathbf{n}\Delta$ of scaled BTW rules (thereby saturating our fixed grayscale palette). (c)–(f) The sandpile relaxes via *slightly* different toppling rules Δ' to an unrecognizable attractor.

is then passed to the site on the right, destroying the symmetry of the original Δ.) Although this is only a slight modification, the final state does not resemble the original image. Since Δ can in general be highly nontrivial, with significant amounts of nonlocal communication, it may be possible to exploit the cryptographic properties of this phenomenon to produce an efficient and effective encryption scheme. In particular, the decoding could be done in parallel via distributed processors, precisely because the Abelian nature of the toppling makes the order of toppling irrelevant. Furthermore, the transformation matrix Δ could be encoded at the hardware level. Fast distributed processing in hardware makes straightforward evolution a practical alternative to implementing a truly a vast lookup table connecting inputs and outputs for the sandpile automaton considered as a finite-state machine.

VII. SUPERPILES

Self-erasing perturbations are not confined to Abelian sandpile automata. For example, there exists a class of related cellular automata for which the addition of any combination of positive *point* perturbations is self-erasing. For every sandpile $\mathbf{P}$, define the corresponding *superpile* $\mathbf{Q}$ such that $\mathbf{Q}=\mathbf{P}\Delta^{-1}$. Mirroring the sandpile toppling $P_i=\mathbf{P}\mathbf{e}_i^T>C_i \Rightarrow T(\mathbf{P})=\mathbf{P}-\mathbf{e}_i\Delta$, define the superpile toppling $\mathbf{Q}\Delta\mathbf{e}_i^T>C_i \Rightarrow T(\mathbf{Q})=\mathbf{Q}-\mathbf{e}_i$. Thus for every self-erasing perturbation $\mathbf{n}\Delta$ of the sandpile, there is a corresponding self-erasing perturbation $\mathbf{n}$ of the superpile. Therefore a superpile attractor is robust to *all* positive perturbations—making them more promising candidates for the robust encoding of information than sandpiles. Although the superpile thresholding may appear complicated, it is easy to describe the BTW superpile toppling: if the discrete Laplacian at a site exceeds the critical value, reduce the site by one. Furthermore, we expect that self-erasing perturbations exist for generalized Abelian sandpiles, such as those studied by Chau and Cheng [12].

VIII. CONCLUSIONS

By generalizing the traditional seeding of Abelian sandpile automata, we have revealed an interesting phenomenon, the existence of a broad and flexible class of nontrivial self-erasing perturbations. We have shown how the conceptual framework provided by SEP is useful in the study of these automata. Furthermore, our demonstration of another automaton that also displays SEP's suggests that this phenomenon may prove more broadly applicable.

More practically, SEP's might be used to secure information. SEP's provide a mechanism by which data might be protected from certain types of noise through the self-healing properties of the sandpile attractors. By recognizing that the addition of noise can be an encryption mechanism, this ability to self-heal becomes an ability to decode encrypted information, provided one has the correct toppling rules Δ, the key to the encryption. More speculatively, if practical physical analogs could be found, this phenomenon might enable the construction of machines that are impervious to normal wear and tear. The operation of such machines would be dynamical attractors whose basin of attraction would include all perturbations away from nominal performance.

ACKNOWLEDGMENTS

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